**Laplace Transformation**

**Definition:** Let the function be defined for all positive values of, then multiply by and integrate it with respect to from zero to infinity. If the resulting integral exists (i.e., has some finite value), it is a function of, may be real or complex, say.

This function of variable is called Laplace Transformation of the original function and will be denoted by, where denotes the Laplace transform operator. Thus

The original function is called the inverse transform or inverse of and will be denoted by.

**Important formulae:**

|  |  |
| --- | --- |
|  | **Some important formulae** |
| , when |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

**Properties of Laplace transformation:**

1. **(linearity)**, where & are constants.

2. If then **(first shifting or translation)**

3. If then **(multiplication by )**

**Proof of some selective formulae:**

**Proof**: From the definition of Laplace transformation we know that,

.

2. and 3. .

**Proof:** - - - - - (1)

But,

So,

- - - -- - (2)

Comparing (1) and (2), we have on equating real and imaginary parts,

and.

**Some workout examples on Laplace transformation:**

|  |  |
| --- | --- |
| **Example: 1** | **Example: 2** |
| **Example: 3**  . | **Example: 4** |
| **Example: 5** | **Example: 6**  . |
| **Example: 7**  (first Shifting Property)  **Or**  **Piecewise function** | **Example: 8**  (first shifting property)  Or |
| **Example:**  Sketch and  find.  **Solution:**  . |  |

**Reference Books:**

1. Advanced Engineering Mathematics- Erwin Kreyszig.(10th Edition)

2. Differential Equations- Paul Blanchard, Robert L. Devaney, Glen R. Hall (4th Edition)

**Problem Set 1**

**Find the Laplace Transforms and also sketch (if free hand sketching is getting complex then use MATLAB) the following functions (1-20):**

**Using direct formula**

1. **Ans:**

2. **Ans:**

3. **Ans:**

4. **Ans:**

5. **Ans:**

6. **Ans:**

7. **Ans:**

8. **Ans:**

**First shifting or Translation property**

If then

9. **Ans:**

10. **Ans:**

11. **Ans:**

12. **Ans:**

13. **Ans:**

**Property of multiplication by**

If then

14. **Ans:**

15. **Ans:**

**Piece-wise Function**

If is a piece-wise function then

|  |  |
| --- | --- |
| 16.  **Ans:** | 17.  **Ans:** |
| 18.  **Ans:**  20.  Also sketch  **Ans:** | 19.  **Ans:** |

**The Unit step function (Heaviside function):**

In engineering applications, we frequently encounter functions whose values change abruptly at specified values of time *t*. One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time *t*.

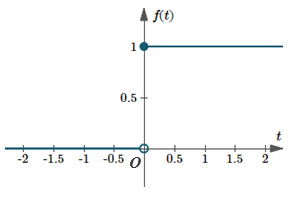
The value of *t* = 0 is usually taken as a convenient time to switch on or off the given voltage.

The switching process can be described mathematically by the function called the **Unit Step Function** (otherwise known as the **Heaviside function** after Oliver Heaviside).

The Unit step function or Heaviside’s unit step functionis defined as follows:

,

That is, is a function of time *t*, and *u* has value **zero** when time is negative (before we flip the switch); and value **one** when time is positive or zero (from when we flip the switch).



Graph of the unit step function.

**Shifted (Right) Unit step function:**

In many circuits, waveforms are applied at specified intervals other than t=0. Such a function may be described using the **shifted** (aka **delayed**) unit step function.

A function which has value0 up to the time and thereafter has value 1 is written:

,

1

O

a

t

**Rectangular pulse:**

A common situation in a circuit is for a voltage to be applied at a particular time (say *t = a*) and removed later, at *t = b* (say). We write such a situation using unit step functions as:

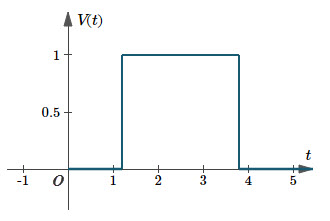
This voltage has strength 1, duration

Alternatively, may be constructed using top hat function as follows:

,

**Example01 :**

The graph of is as follows. Here, the duration is 3.8−1.2=2.6 .



Graph ofis an example of a rectangular pulse.

**Example 02:**

Write the following functions in terms of **unit step** function(s). Sketch each waveform.

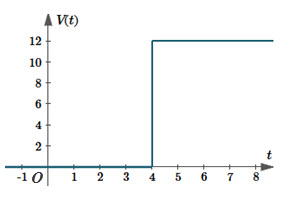
1. A 12-V source is switched on at *t* = 4 s,
2. and,
3. .

**Solution:**

**(i)**Since the voltage is turned on at t = 4, we need to use We multiply by 12 since that is the voltage.

We write the function as follows:

Here's the graph:



Graph of

**(ii)**,

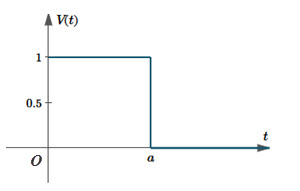
In words, the voltage has value 1until time. Then it is turned off.

We have a "rectangular pulse" situation and need to use this formula:

In our example, the pulse starts at , so we use and finishes at so we use

So the required function is:

Here is the graph



Graph of, a shifted unit step function

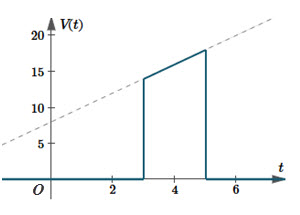
**(iii)**

In this example, our function is which has slope 2 and V-intercept 8.

The signal is only turned on between and. The rest of the time it is off.

So our voltage function will be:

The graph is as follows:



Graph of. The dashed line is

Now, the Laplace transform of **unit step function** is,

**Example: 01**

Find the Laplace transformation of

**Solution:**

**Example: 02**

Find the Laplace transformation of

**Solution:**  .

**Example: 03**

Find the Laplace transformation of

**Solution:**

**Example: 04**Given,

1. sketch ,
2. convert to unit step function and,
3. find the Laplace transformation of.

**Solution:**

|  |  |
| --- | --- |
|  |  |

**Example:05**Givenwhere E, *a* and *b* are positive constants.

1. sketch ,
2. convert to unit step function and,
3. find the Laplace transformation of.

**Solution:**

|  |  |
| --- | --- |
|  | *=* |

**Example: 06**Given

1. sketch ,
2. convert to unit step function and,
3. find the Laplace transformation of.

**Solution:**

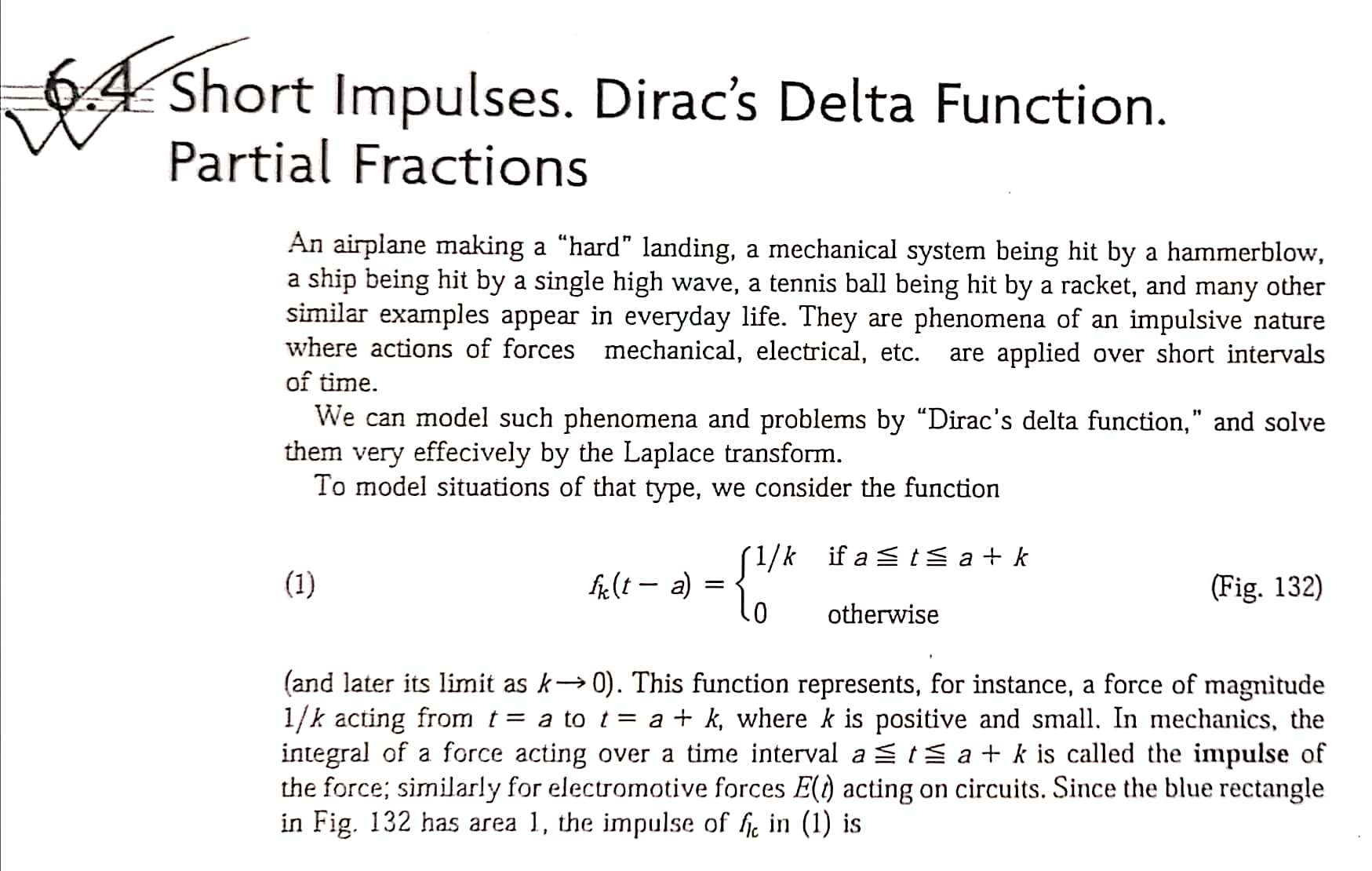
|  |  |
| --- | --- |
|  |  |

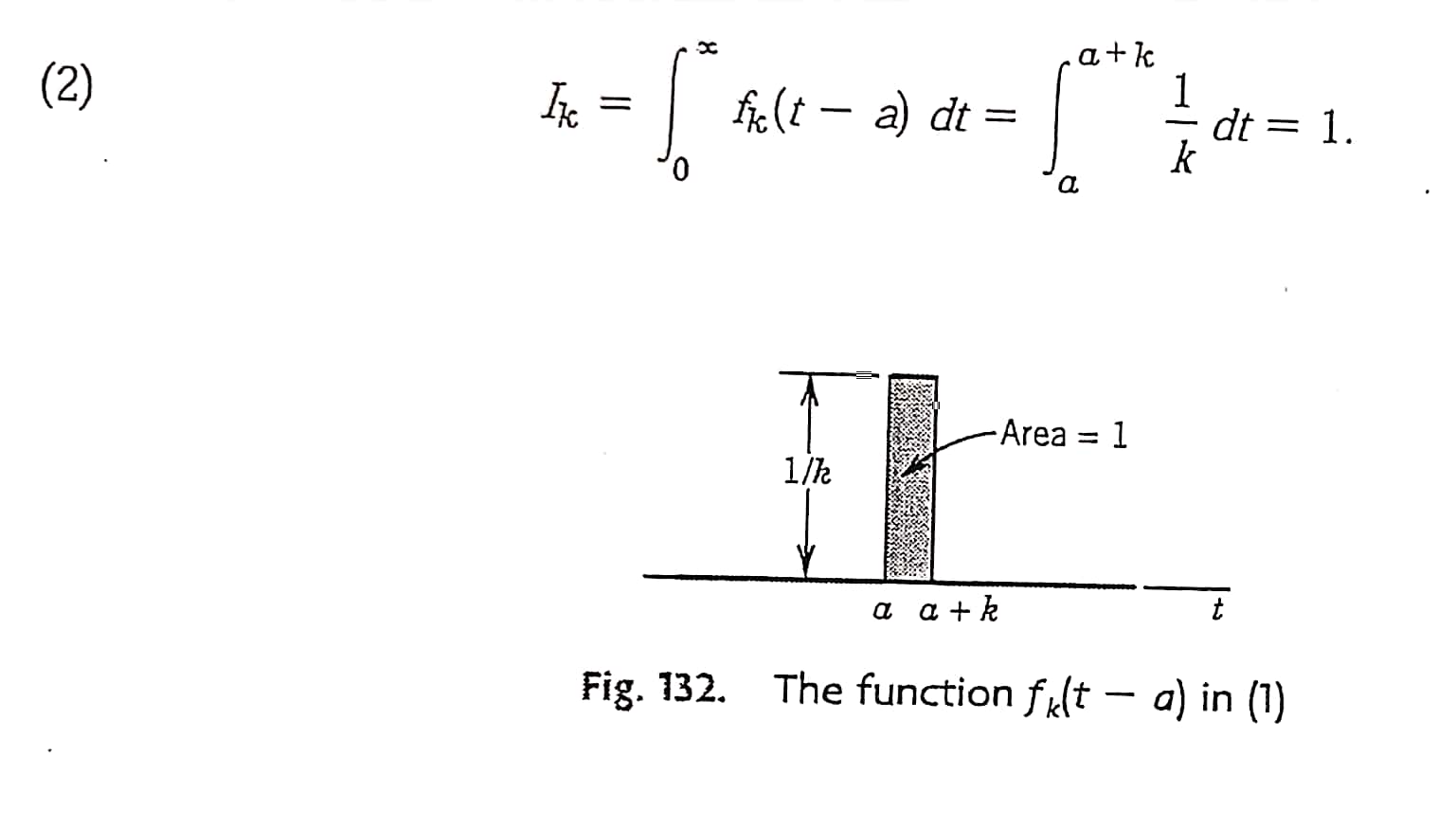
**Example: 07**Given,

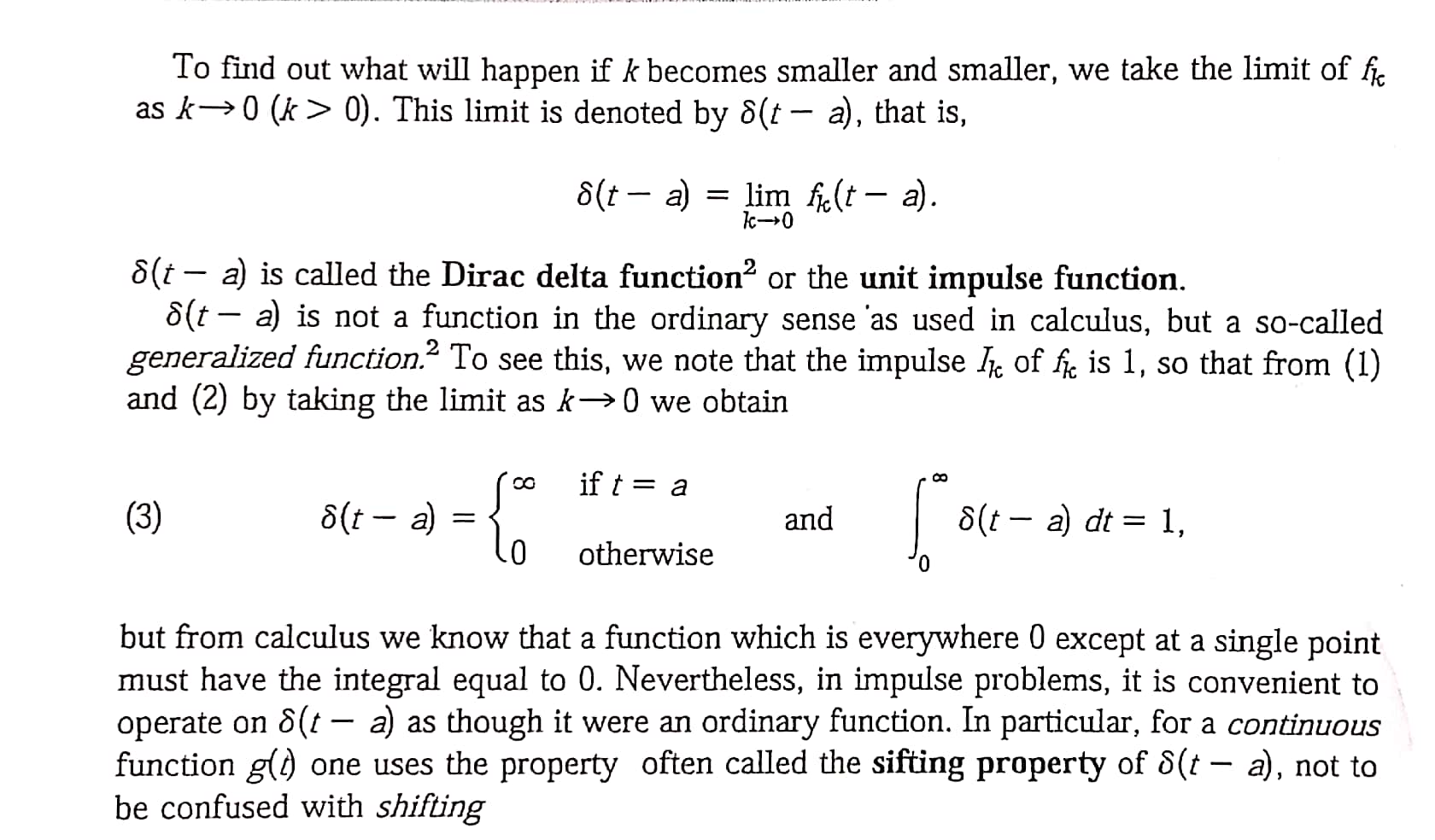
1. sketch ,
2. convert to unit step function and,
3. find the Laplace transformation of.

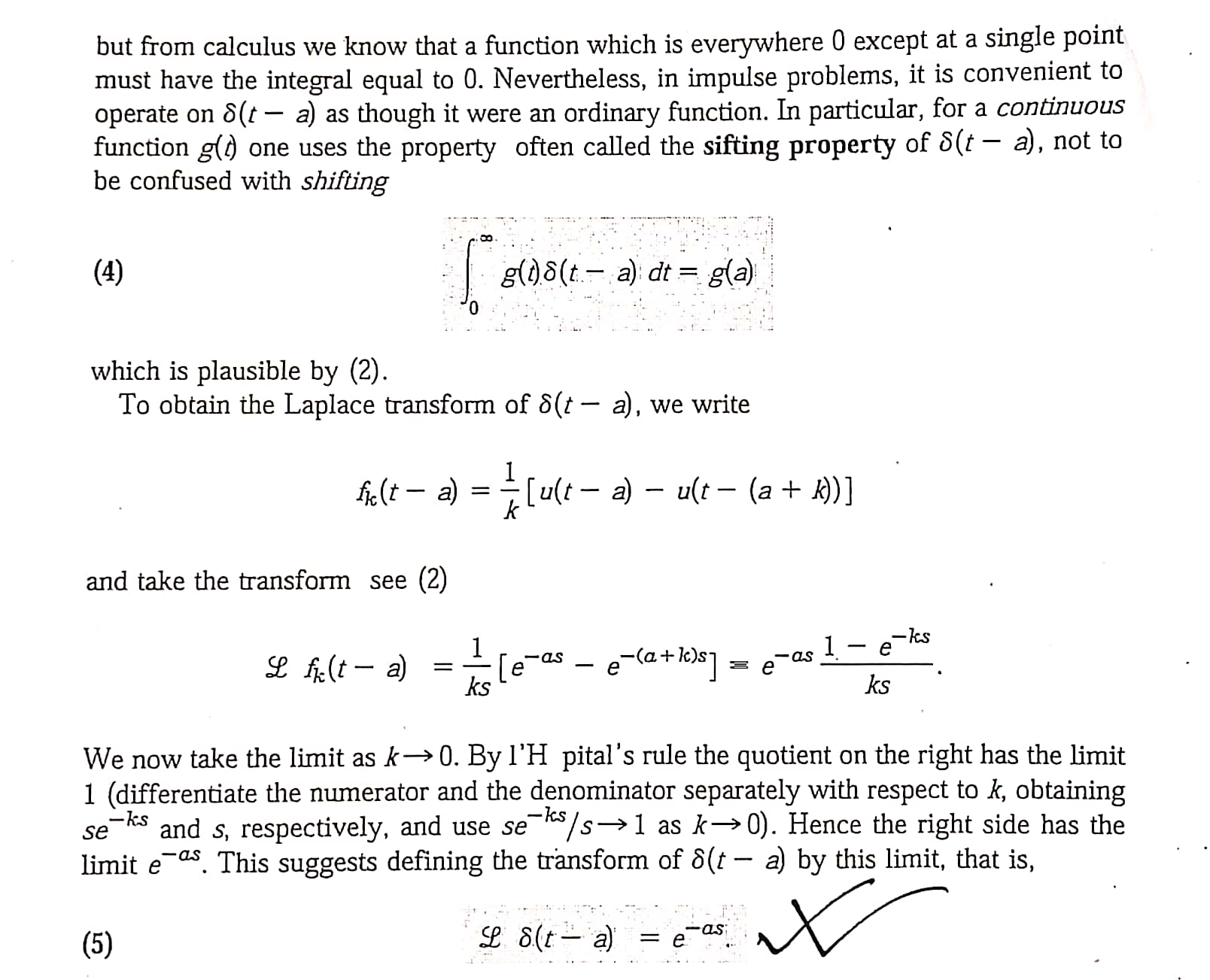
**Solution:**

|  |  |
| --- | --- |
|  | . |

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**Problem Set 2**

**Unit step function**

**Sketch the following functions and find their Laplace transforms (21-25):**

21. **Ans:**

22. **Ans:**

23. **Ans:**

24. **Ans**:

25. **Ans:**

**Sketch the following functions. Write in terms of unit step function and hence find**

**their Laplace transforms: (26-27)**

26.

**Ans:**

27. ,

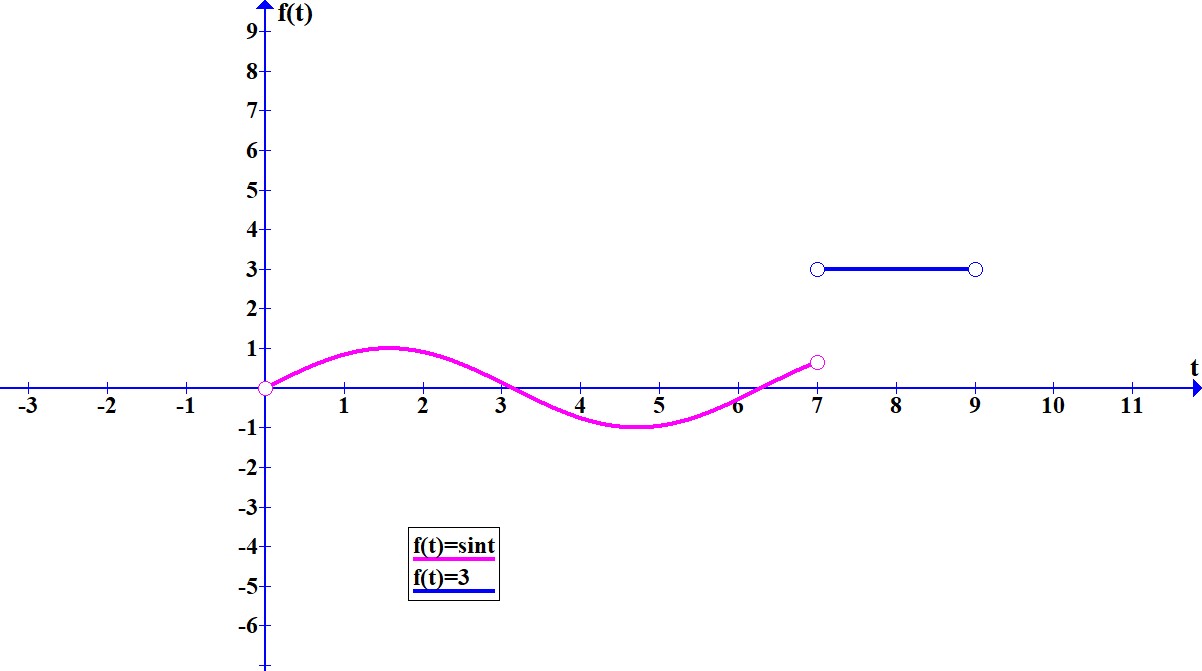
**Ans:**

MATLAB answer.  
 By hand calculation. But both expressions are

same.

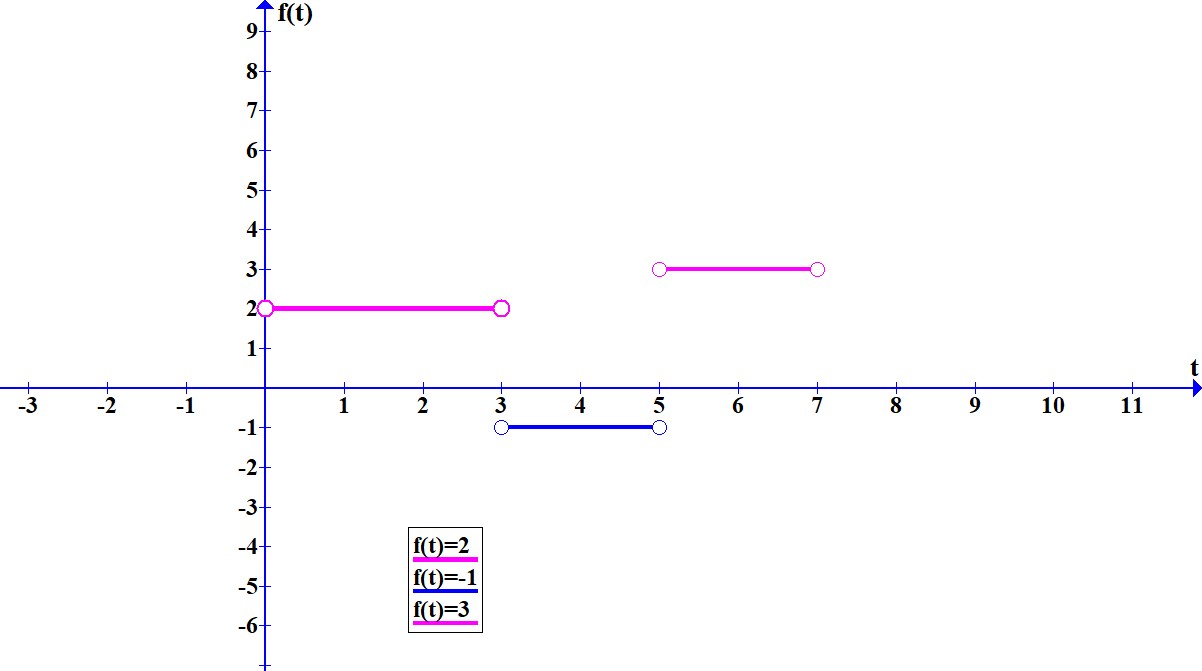
**Using the following graphs write in terms of unit step function and hence find the Laplace transforms of (28-29)**

28.



**Ans:**

29.



**Ans:**

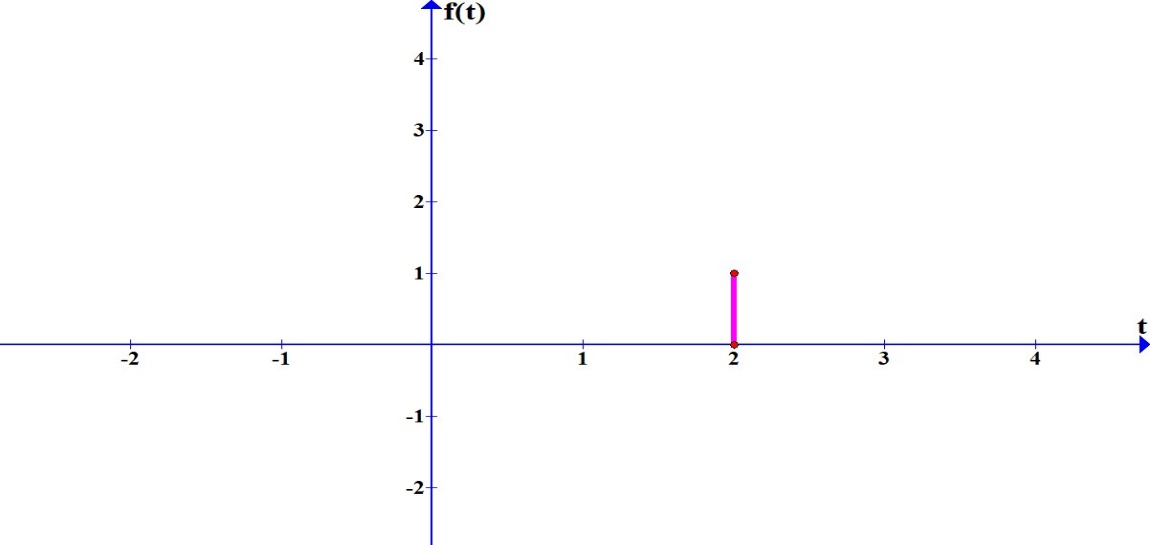
**Dirac’s delta (Unit impulse) function**

**Sketch the following functions and find their Laplace transforms: (30-32)**

.

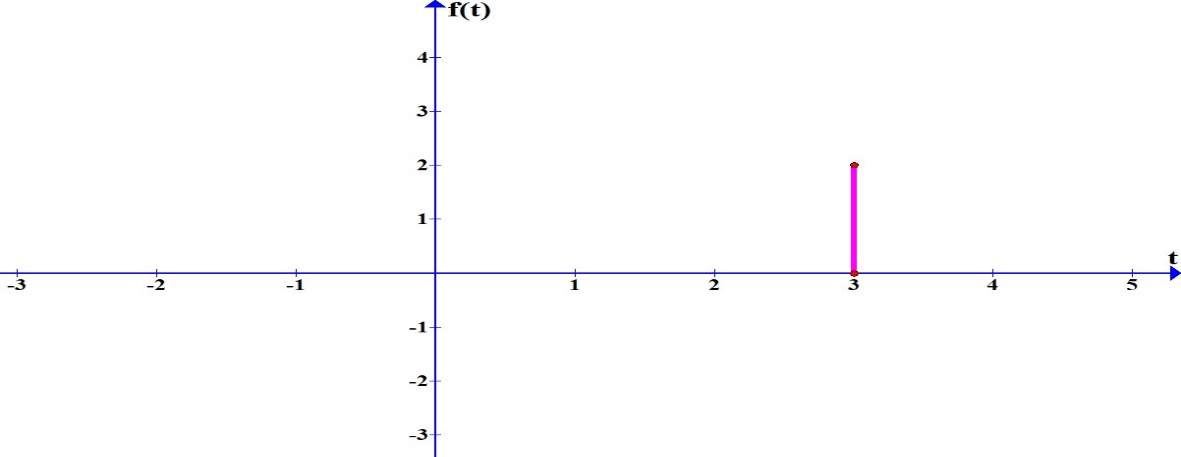
30.

**Ans:**



31.

**Ans:**



32.

**Ans:**

